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Evaluating the Risk of Urban Redevelopment Viewed as a Social Decision Process

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Abstract Under the scheme of Japanese City Renewal Law (JCRL), the cost of urban redevelopment project has been financed by the revenue from selling the reservation floor secured beforehand in the redeveloped building. Since the outcome of whether or not the reservation floor can be sold is unforeseeable, entities for redevelopment are faced with the risk for project financing. Traditional risk evaluation has just focused on the risk inherent in the project that is carried out by a single entity and whose future returns are uncertain. On the contrary, this paper focuses on the risk of urban redevelopment project in which plural decision-makers are involved and whose future outcome becomes uncertain since it depends on the decisions made by other decision-makers involved. Defining the risk for the entities as the probability that the entities suffer a loss and formalizing the urban redevelopment procedure as a two-stage auction game, this paper has demonstrated that we can evaluate the actual amount of risk the entities are going to bear in the project by showing a hypothetical numerical example.

Keywords Risk Evaluation, Urban Redevelopment Procedure, Game Theory, Auction Model

1 Introduction

Under the scheme of Japanese City Renewal Law (JCRL), the cost of urban redevelopment project has been financed by the revenue from selling the reservation floor secured beforehand in the redeveloped building. Since the outcome of whether or not the reservation floor can be sold is unforeseeable, entities for redevelopment are faced with the risk for project financing.

As seen in financial engineering studies such as DDCF (Dynamic Discounted Cash Flow) and real options (See for example, Kariya, Ohara, Honkawa (2002) [2], Kawaguchi (2002) [3], Trigeorgis (1996) [7]), researches on risk evaluation traditionally have been focusing on the risk entailed by the project that is carried out by a single entity and whose future returns are uncertain. On the contrary, this paper focuses on the risk of urban redevelopment project in which plural decision-makers are involved and whose future outcome becomes uncertain since it depends on the decisions made by other decision-makers involved.

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Defining the risk for the entities as the probability that the entities suffer a loss and formalizing the urban redevelopment procedure as a two-stage auction game, this paper aims to demonstrate that we can evaluate the actual amount of risk the entities are going to bear in the project by showing a hypothetical numerical example.

The game theoretic model for urban redevelopment procedure was first constructed by Imanishi, Saito and Tanaka (2006) [1]. They have addressed the problem to evaluate what amount of risk would be transferred from UR (Urban Renaissance Agency), a government entity for redevelopment authorized by JCRL, to the private sector if UR employs a new contract scheme under which UR can delegate the disposition of the reservation floor to the constructor who makes a contract with UR to construct the redeveloped building.

Following their model, our model also is formalized as a two-stage auction game in which the first auction is performed to choose a constructor for the redeveloped building as a sealed bid lowest price auction and the second one to choose a buyer for the reservation floor as a sealed bid highest price auction. In Imanishi, Saito and Tanaka (2006) [1], they theoretically derived the risk born by the entities for the redevelopment under a hypothetical case where true preferences of buyers and constructors are distributed according to uniform distributions. While their derivation is an innovative one, they have derived the result for just one case out of all four possible cases. In this paper we have shown derivations for all these four cases.

The remaining parts of this paper are composed as follows. In Chapter 2 we introduce our model. Theoretical derivations of the risk under uniform distribution are shown in Chapter 3. Chapter 4 gives numerical simulation examples. We conclude in Chapter 5.

2 Model

We consider an urban redevelopment project that plans to build a high-rise redeveloped building that includes the reservation floor to be sold for compensating the project cost. In this traditional urban redevelopment project, three kinds of players are involved: (1) The entity that carries out the project taking the risk of project financing, (hereafter, the redevelopment entity), (2) A building constructor who constructs the redeveloped building, and (3) A buyer who buys the reservation floor.

The process of a typical urban redevelopment project proceeds as follows:

1. The redevelopment entity performs the first-stage auction to choose among the bidders a constructor to build the redeveloped building in the form of the first lowest price sealed-bid auction.
2. The bidder who set the lowest price is chosen as the constructor.
3. The redevelopment entity performs the second-stage auction to choose among the bidders a buyer to buy the reservation floor in the form of the first highest price sealed-bid auction.
4. The bidder who set the highest price is chosen as the buyer of the floor.
5. The payoffs are determined for all three players: the redevelopment entity implementing the project, the constructor, and the reservation floor buyer.

We formulate this procedure as a two-stage auction game in the following way. Let $M = \{1, 2, ..., m\}$ and $N = \{1, 2, ..., n\}$ be respectively the set of bidders in the auction for the reservation floor and the set of bidders in the auction for the building construction indexed as $i \in M$ and $j \in N$. We denote the payoff for the redevelopment entity by u_0, the payoffs for $i \in M$ and $j \in N$ by u_i and u_j. Let $e_i > 0$ and $c_j > 0$ denote the true values of the reservation floor and the building construction cost for bidder i and j respectively.

We make the following assumptions. First, every player is assumed risk neutral. Next, every bidder for each auction is symmetric. In other words, e_i for every i is distributed according to the same distribution F and c_j for every j according to the same distribution G. Third, for the bidding process of the reservation floor auctions, all bidders are assumed to know their own true value e_i, but do not know the true values for all other bidders. Hence, each bidder i regards all bidding prices $e_{i'}$ charged by all other bidders i' as though they were independently drawn from distribution F. We assume that F has a differentiable density with the support of $[\underline{e}, \overline{e}]$.

In the same way, for the bidding process of construction auction, all bidders are assumed to know their own true value c_j, but do not know the true values for other bidders. Hence, each bidder j regards all bidding prices $c_{j'}$ charged by all other bidders j' as though they were independently drawn from distribution G. We also assume that G has a differentiable density with the support of $[\underline{c}, \overline{c}]$. These assumptions are typical for the discussion of Bayesian Nash equilibrium. Finally, we assume that two auctions be carried out independently.

From these, the urban redevelopment process can be described as follows. First we draw at random $c_j (j \in N)$ for all bidders of constructors from G. Next, they decide their own bidding prices $b_j (j \in N)$. Then the successful bidder j^* and his bidding price b_{j^*} are determined. Similarly, we draw at random $e_i (i \in M)$ for all bidders of reservation floor buyers from F. Then they decide their own bidding prices $t_i (i \in M)$. Then the successful bidder i^* and his bidding price t_{i^*} are determined.

At the final step, the payoffs for all three players are determined. For the constructors, $u_{j^*} = b_{j^*} - c_{j^*}$ if j is the successful bidder, i.e., $j = j^*$ and $u_j = 0$, otherwise, i.e., $j \neq j^*, j \in N$, for the reservation floor buyers, $u_{i^*} = e_{i^*} - t_{i^*}$ if i is the successful bidder, i.e., $i = i^*$ and $u_i = 0$, otherwise, i.e., $i \neq i^*, i \in M$, and the payoff for the redevelopment entity becomes $u_0 = t_{i^*} - b_{j^*}$.

It is well known that for the auction game with incomplete information under Bayesian Nash assumptions the optimal strategy for each participant is derived as a closed form. We state the optimal strategy for each bidder for the above two independent auctions as the following propositions.

Proposition 1. *In the auction for the reservation floor, the optimal strategy t_i of the bidder i with value e_i is*

$$t_i = t(e_i) = e_i - \int_{\underline{e}}^{e_i} \left(\frac{F(y)}{F(e_i)} \right)^{m-1} dy. \tag{1}$$

Proposition 2. *In the auction for the building construction contract, the optimal strategy b_j of the bidder j with value c_j is*

$$b_j = b(c_j) = c_j + \int_{c_j}^{\overline{c}} \left(\frac{1 - G(x)}{1 - G(c_j)} \right)^{n-1} dx. \tag{2}$$

For the proof of these propositions, refer to Myerson (1991) [6], Miura (2003) [5], and Imanishi, Saito and Tanaka (2006) [1].

3 The Risk of Urban Redevelopment Procedure

We define the risk of urban redevelopment procedure as the probability that the redevelopment entity suffers the negative profit from the redevelopment implementation process.

Based on this definition, we wish to calculate the risk of urban redevelopment procedure under the specific distributional assumption that F and G are distributed as uniform distributions on $[\underline{e}, \overline{e}]$ and $[\underline{c}, \overline{c}]$ respectively.

Under these assumptions, a simple calculation from the above propositions leads to the following.

$$b_j = b(c_j) = \frac{(n-1)\,c_j + \overline{c}}{n}, \quad t_i = t(e_i) = \frac{(m-1)\,e_i + \underline{e}}{m} \tag{3}$$

$$\underline{b} = \frac{(n-1)\underline{c} + \overline{c}}{n}, \quad \overline{b} = \overline{c}, \quad \underline{t} = \underline{e}, \quad \overline{t} = \frac{(m-1)\overline{e} + \underline{e}}{m} \tag{4}$$

Hence both distributions for b_j and t_i become uniform distributions. Let K be the survival function for $b_j = b(c_j)$ and H the distribution function for $t_i = t(e_i)$. We obtain

$$K(x) = \frac{\overline{b} - x}{\overline{b} - \underline{b}}, (x \in [\underline{b}, \overline{b}] \subset R_+), \quad H(y) = \frac{y - \underline{t}}{\overline{t} - \underline{t}}, \text{ (y} \in [\underline{t}, \overline{t}] \subset R_+) \tag{5}$$

Denoting by k and h the density of K and H, the probability that the payoff of the redevelopment entity becomes negative can be expressed as follows.

$$I = \iint_S k(x)h(y)K(x)^{n-1}H(y)^{m-1}dxdy = A\iint_S (\overline{b} - x)^{n-1}(y - \underline{t})^{m-1}dxdy, \tag{6}$$

where $S = \{(x,y) \in [\underline{b}, \overline{b}] \times [\underline{t}, \overline{t}] | x \geq y\} \subset R_+ \times R_+$ and $A = 1/((\overline{b} - \underline{b})^n(\overline{t} - \underline{t})^m)$.

The integrand of the right hand side of the first equation in Equation (6) is interpreted as follows. The term of $K(x)^{n-1}H(y)^{m-1}$means the conditional probability that given (x,y), both of bidding prices x and y become successful bids. The term of $k(x)h(y)$ is the probability to draw the independent sample (x,y) from K and H so that the integrand is equal to the probability that the bidding prices (x,y) become both successful bids. Note that the area S means the building construction cost x is equal to or greater than the reservation floor price$y$. Thus the integration of Equation (6) is identical to the definition of the risk of urban redevelopment procedure.

Suppose that when the redevelopment entity encounters the negative payoff he would repeat auctions until his payoff becomes positive. Then we can use the number of auction repetitions required to get a positive profit as an index of the redevelopment risk.

Let $p(m,n)$ denote the probability that the entity's payoff becomes negative. Define the random variable $X(m,n)$ as the number of auction repetitions required for the

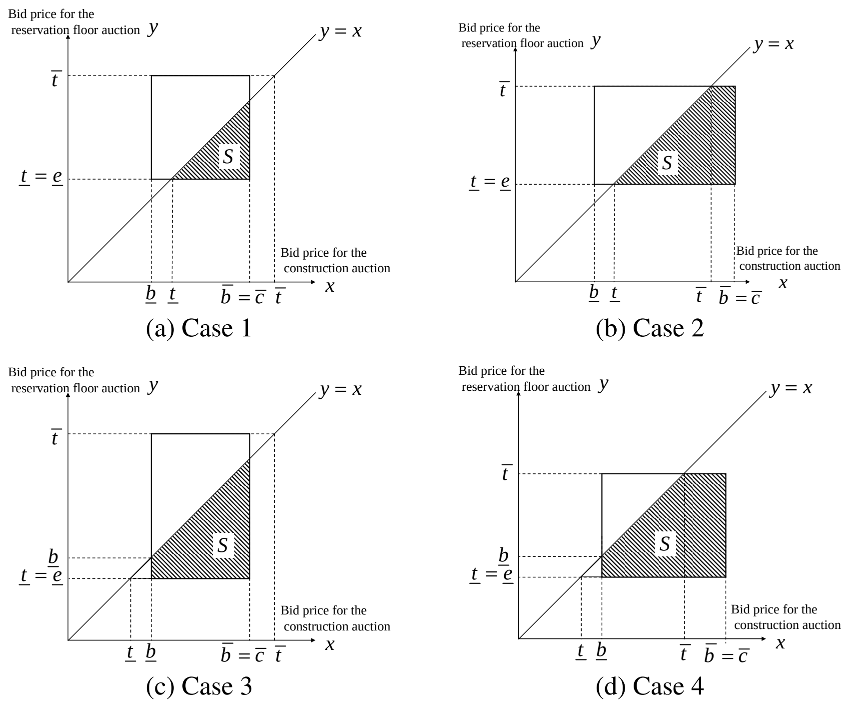

Figure 1: Four cases

redevelopment entity to reach a positive payoff for the first time. Note that the probability that entity's payoff becomes positive for the first time at the kth repetition is $p(m,n)^{k-1}(1-p(m,n))$. Thus $X(m,n)$ is distributed as geometric distribution. Hence the expected number of minimum auction repetitions required to reach positive payoff for the first time becomes as follow.

$$\frac{1}{1-p(m,n)} \tag{7}$$

With these setups, we can calculate the risk of urban redevelopment procedure.

To calculate the integration of Equation (6), it is convenient to classify the area S into four cases depending on the value of $\underline{t}$,$\bar{t}$,$\underline{b}$, and $\overline{b}$ as depicted in Figure 1.

We provide the results for these four cases below.

Case 1 ($\underline{b} \le \underline{t} \le \overline{b} \le \bar{t}$)

Figure 1 (a) shows the area where the entity's payoff becomes negative in $\underline{b} \le \underline{t} \le \overline{b} \le \bar{t}$. In this case, the risk is

$$I = \frac{A}{mn} \cdot \frac{1}{{}_{n+m}C_m} (\overline{b} - \underline{t})^{n+m} \tag{8}$$

Case 2 ($\underline{b} \leq \underline{t} \leq \bar{t} \leq \overline{b}$)

Figure 1 (b) shows the area where the entity's payoff becomes negative in $\underline{b} \leq \underline{t} \leq \bar{t} \leq \overline{b}$. In this case, the risk is

$$I = \frac{A}{nm} \cdot \frac{1}{{}_{n+m}C_m}((\overline{b} - \underline{t})^{n+m} - (\overline{b} - \bar{t})^{n+m}) - \frac{A}{mn} \sum_{k=1}^{m-1} \frac{n!m!}{(m-k)!(n+k)!} S_1(k), \tag{9}$$

where $S_1(k) = (\overline{b} - \bar{t})^{n+k}(\bar{t} - \underline{t})^{m-k}$

Case 3 ($\underline{b} \geq \underline{t}, \overline{b} \leq \bar{t}$)

Figure 1 (c) shows the area where the entity's payoff becomes negative in $\underline{b} \geq \underline{t}, \overline{b} \leq \bar{t}$. In this case, the risk is

$$I = \frac{A}{nm} \left((\overline{b} - \underline{b})^n (\underline{b} - \underline{t})^m + \frac{1}{{}_{n+m}C_m} (\overline{b} - \underline{b})^{n+m} \right) + \frac{A}{mn} \sum_{k=1}^{m-1} \frac{n!m!}{(m-k)!(n+k)!} S_2(k), \tag{10}$$

where $S_2(k) = (\overline{b} - \underline{b})^{n+k}(\underline{b} - \underline{t})^{m-k}$

Case 4 ($\underline{b} \geq \underline{t}, \overline{b} \geq \bar{t}$)

Figure 1 (d) shows the area where the entity's payoff becomes negative in $\underline{b} \geq \underline{t}, \overline{b} \geq \bar{t}$. In this case, the risk is

$$I = \frac{A}{nm} (\overline{b} - \underline{b})^n (\underline{b} - \underline{t})^m - \frac{A}{mn} \sum_{k=1}^{m-1} \frac{n!m!}{(m-k)!(n+k)!} (S_1(k) - S_2(k)) \tag{11}$$

4 Numerical Examples

In Table 1 we provide numerical examples corresponding to the above four cases. Look at Table 1(d) for instance. The caption above this table tells that the true distribution of building construction cost is uniform distribution on $[\underline{c}, \overline{c}] = [1, 3]$ and that of reservation floor value is uniform distribution on $[\underline{e}, \overline{e}] = [1, 3.2]$. Each of 16, 4 by 4 cells corresponds to different combinations with different numbers of m and n, where m is the number of floor auction bidders and n is the number of construction auction bidders. The figures in each cell shows the expected minimum number of auction repetitions to get the positive payoff, $E(X(m,n))$, which is equal to $1/(1 - p(m,n))$. Each row of the additional two columns attached to the left side of the table contains the numbers of $[\underline{b}, \overline{b}]$, the minimum and the maximum of construction bidding prices calculated from n, $\underline{c}$, and $\overline{c}$. Similarly each column of the additional two rows above the table contains the numbers of $[\underline{t}, \bar{t}]$, the minimum and the maximum of the reservation floor bidding prices calculated from m, $\underline{e}$, and $\overline{e}$. From these, we see that all 16 cells fall into Case 4 because they all satisfy the condition, $\underline{b} \geq \underline{t}$ and $\overline{b} \geq \bar{t}$.

From these four tales, we see the expected values of true evaluation of construction costs for bidders are 2 for all 4 cases. On the other hand, the expected values of true evaluation of the reservation floor for bidders are 3, 2.1, 3, and 2.1 for Case 1 to Case 4 respectively. Therefore, if true evaluations were tendered by bidders, the expected payoff

Table 1: The expected number of auction repetitions required for the redevelopment entity to get the positive payoff

(a) Case 1

$\underline{c}=1, \overline{c}=3, \underline{e}=2, \overline{e}=4$

		$\overline{t}$	3	3.3	3.5	3.6
		$\underline{t}=\underline{e}$	2	2	2	2
$\underline{b}$	$\overline{b}=3$	n \ m	2	3	4	5
2	3	2	1.20	1.04	1.01	1.00
1.67	3	3	1.04	1.01	1.00	1.00
1.5	3	4	1.01	1.00	1.00	1.00
1.4	3	5	1.00	1.00	1.00	1.00

(b) Case 2

$\underline{c}=1, \overline{c}=3, \underline{e}=2, \overline{e}=2.2$

		$\overline{t}$	2.1	2.13	2.15	2.16
		$\underline{t}=\underline{e}$	2	2	2	2
$\underline{b}$	$\overline{b}=3$	n \ m	2	3	4	5
2	3	2	7.79	5.28	4.44	4.03
1.67	3	3	1.52	1.45	1.40	1.38
1.54	3	4	1.18	1.15	1.14	1.13
1.4	3	5	1.07	1.06	1.05	1.05

(c) Case 3

$\underline{c}=1, \overline{c}=3, \underline{e}=0.5, \overline{e}=5.5$

		$\overline{t}$	3	3.38	4.25	4.5
		$\underline{t}=\underline{e}$	0.5	0.5	0.5	0.5
$\underline{b}$	$\overline{b}=3$	n \ m	2	3	4	5
2	3	2	2.21	1.21	1.07	1.02
1.67	3	3	1.59	1.11	1.03	1.01
1.54	3	4	1.39	1.07	1.02	1.01
1.4	3	5	1.29	1.05	1.01	1.00

(d) Case 4

$\underline{c}=1, \overline{c}=3, \underline{e}=1, \overline{e}=3.2$

		$\overline{t}$	2.1	2.47	2.65	2.76
		$\underline{t}=\underline{e}$	1	1	1	1
$\underline{b}$	$\overline{b}=3$	n \ m	2	3	4	5
2	3	2	15.53	3.03	1.86	1.47
1.67	3	3	2.56	1.49	1.22	1.12
1.54	3	4	1.70	1.22	1.09	1.05
1.4	3	5	1.41	1.12	1.05	1.02

for the redevelopment entity should be 1.0, 0.1, 1.0, and 0.1 for case 1 to case 4 respectively. However, while Case 1 (Case 2) and Case 3 (Case 4) have the same expected profit the risk represented by the number of auction repetitions turn out to be quite different. In fact, for the case of n=2 and m=2, the expected numbers of auction repetitions vary from 1.20 (7.79) to 2.21 (15.53) from Case 1 (Case 2) to Case 3 (Case 4). These facts imply that the risks born by the redevelopment entity are quite different even though the expected profits are the same.

Thus, we have demonstrated that expressing the risk by the expected minimum number of the auction repetitions required to arrive at the positive profit should be very effective for the redevelopment entity to understand the risk of urban redevelopment process it plans to organize.

Our risk index by the number of auction repetitions can be transformed into money terms. We will show this possibility with providing an example based on the actual instance. We take up the urban redevelopment project carried out in a city center area of Fukuoka City 10 years ago. The project plans to construct the high-rise redeveloped buildings on the site secured by removing low-rise small shops and houses densely located there. The project enforcement entity is the cooperative formed by land owners. The developer, Fukuoka City Futures (FCF) Inc., a third sector subsidiary of Fukuoka City Government participated in this project as an agent for the redevelopment enforcement cooperative.

The planned redeveloped buildings contained the large size of reservation floors, a hotel, a fine art museum, a shopping mall, and so on. The total cost of the project was 97.8 billion yen. The reservation floors were sold at the price of 90.0 billion yen. Among them, Hotel Okura Fukuoka bought the hotel floor at the price of 22.9 billion yen, and SBC Inc., a subsidiary of FCF, bought the floor for the shopping mall at the price of 56.7

billion yen, which was finance by borrowing from Local Banks. The project also got the government subsidy of 8.0 billion yen. However, SBC went into bankruptcy in 2002 and SBC has been liquidated after Local Banks gave debt forgiveness of 45.0 billion yen. Also Hotel Okura went into trouble but was rehabilitated by obtaining from Local Banks the debt forgiveness of 10.0 billion yen.

Now let us apply our model to this case. Assume that the cost of one repetition of auction is 1/10 of the total cost of the project. Suppose that true evaluation of building construction cost for the constructors is distributed uniformly on the interval between 80.0 and 100.0 billion yen. Suppose also that true evaluation of the reservation floor for the buyers is distributed uniformly on the interval between 90.0 and 92.0 billion yen. Assume n=2 and m=2. In short, we have assumed that $[\underline{c}, \overline{c}] = [80.0, 100.0]$, $[\underline{e}, \overline{e}] = [90.0, 92.0]$ for $m = n = 2$. From this, we see this case corresponds to Case 2. From Table 1(b), the number of auction repetitions is 7.79 so that the additional cost incurred by auction repetitions becomes 67.9% of the total cost. Thus in this case we see that our model estimates the risk of this project as 66.4(=97.8*0.679) billion yen. This amount becomes almost the same as the actual loss of this project, 63.0 billion yen, that is, the debt waiver of 55.0 billion yen from Local Banks plus 8.0 billion yen of government subsidy.

5 Conclusion

We have formulated the urban redevelopment procedure as a two-stage auction game in which three agents are involved to transact each other: the redevelopment entity that performs auctions, constructors participating in the construction auction, and buyers participating in the reservation floor auction. By defining the risk of the urban redevelopment procedure by the probability that the redevelopment entity suffers a negative profit, we have shown that the risk can theoretically be derived in a closed form and can be calculated numerically under the assumption that preferences of bidders for both auctions are distributed as uniform distributions with giving numerical examples.

In contrast to the traditional researches on risk evaluation such as DDCF, real options, and financial engineering in which a single agent is involved facing with uncertainty inherent in the project, the most significant contribution of this paper, we believe, is that we have formulated for the first time a model that can be used to evaluate quantitatively the risk of the urban redevelopment procedure where plural decision-makers are involved and the amount and distribution of which change depending on the decisions made by other decision-makers. In other words, in a social decision process such like urban redevelopment project, the risk of the project not only depends on the uncertainty inherent in the project but also critically depends on how the decision making processes are organized, what kinds of behaviors participants would take, what decisions they would make, and so on. Thus we need a model that can be utilized for the analysis of these phenomena. Our model tries to be a first step toward such a direction.

Appendix

Derivation of equation (8)

For Case 1, Equation (6) is rewritten as follows.

$$I = A \int_{\underline{t}}^{\overline{b}} (\overline{b} - x)^{n-1} \left(\int_{\underline{t}}^{x} (y - \underline{t})^{m-1} dy \right) dx \tag{12}$$

Let I_1 be the second integration of the right side, and we have

$$I_1 = \frac{1}{m}(x-\underline{t})^m \tag{13}$$

Hence, we have

$$I = \frac{A}{m}\int_{\underline{t}}^{\overline{b}} (\overline{b}-x)^{n-1}(x-\underline{t})^m dx \tag{14}$$

Putting the integration of the right hand side of Equation (14) as $I_2(n-1,m)$ and applying the integration by parts we get the following recursive formula.

$$I_2(n-1,m) = \frac{m}{n}\int_{\underline{t}}^{\overline{b}} (\overline{b}-x)^n (x-\underline{t})^{m-1} dx = \frac{m}{n} I_2(n,m-1) \tag{15}$$

Using Equation (15) recursively, we get

$$I_2(n-1,m) = \frac{m}{n} I_2(n,m-1) = \frac{m(m-1)}{n(n+1)} I_2(n+1,m-2) = \frac{m!(n-1)!}{(n+m-1)!} I_2(n+m-1,0) \tag{16}$$

Since $I_2(n+m-1,0) = \int_{\underline{t}}^{\overline{b}} (\overline{b}-x)^{n+m-1} dx = \frac{1}{n+m}(\overline{b}-\underline{t})^{n+m}$, we have $I_2(n-1,m) = \frac{m!(n-1)!}{(n+m)!}(\overline{b}-\underline{t})^{n+m}$. Equation (8) follows by substitution.

Derivation of equation (9)

For Case 2, Equation (6) is rewritten as follows.

$$I = A\int_{\underline{t}}^{\overline{t}} (\overline{b}-x)^{n-1}\left(\int_{\underline{t}}^{x} (y-\underline{t})^{m-1} dy\right) dx + A\int_{\underline{t}}^{\overline{b}} (\overline{b}-x)^{n-1}\left(\int_{\underline{t}}^{\overline{t}} (y-\underline{t})^{m-1} dy\right) dx \tag{17}$$

Let I_3 be the first integration term of equation of the right hand side. We have

$$I_3 = A\int_{\underline{t}}^{\overline{t}} (\overline{b}-x)^{n-1} I_1 dx = \frac{A}{m}\int_{\underline{t}}^{\overline{t}} (\overline{b}-x)^{n-1}(x-\underline{t})^m dx \tag{18}$$

Applying the integration by parts, we get

$$I_3 = -\frac{A}{mn}(\overline{b}-\overline{t})^n(\overline{t}-\underline{t})^m + \frac{A}{n}\int_{\underline{t}}^{\overline{t}} (\overline{b}-x)^n(x-\underline{t})^{m-1} dx \tag{19}$$

Denote by I_4 the second integration term of the right hand side of Equation (17). Set $I_5 = \int_{\underline{t}}^{\overline{t}} (y-\underline{t})^{m-1} dy$. Since $I_5 = \frac{1}{m}(\overline{t}-\underline{t})^m$, we have $I_4 = \frac{A}{mn}(\overline{b}-\overline{t})^n(\overline{t}-\underline{t})^m$. Thus Equation (17) can be expressed as

$$I = \frac{A}{n}\int_{\underline{t}}^{\overline{t}} (\overline{b}-x)^n(x-\underline{t})^{m-1} dx. \tag{20}$$

Set $I_6(n,m-1) = \int_{\underline{t}}^{\overline{t}} (\overline{b}-x)^n(x-\underline{t})^{m-1} dx$. Then

$$I_6(n,m-1) = \frac{m-1}{n+1}\int_{\underline{t}}^{\overline{t}} (\overline{b}-x)^{n+1}(x-\underline{t})^{m-2} dx - \frac{1}{n+1}(\overline{b}-\overline{t})^{n+1}(\overline{t}-\underline{t})^{m-1} \tag{21}$$

Put $S_1(k) = (\overline{b}-\overline{t})^{n+k}(\overline{t}-\underline{t})^{m-k}$. Using the above recursion repeatedly, we have

$$\begin{aligned} I_6(n,m-1) &= \frac{m-1}{n+1} I_6(n+1,m-2) - \frac{1}{n+1} S_1(1) \\ &= \frac{(m-1)(m-2)}{(n+1)(n+2)} I_6(n+2,m-3) - \frac{(m-1)\cdot 1}{(n+1)(n+2)} S_1(2) - \frac{1}{n+1} S_1(1) \qquad (22) \\ &= \frac{(m-1)!n!}{(n+m-1)!} I_6(n+m-1,0) - \sum_{k=1}^{m-1} \frac{n!(m-1)!}{(m-k)!(n+k)!} S_1(k) \qquad (23) \end{aligned}$$

Noting that $I_6(n+m-1,0) = \frac{1}{n+m}\left((\overline{b}-\underline{t})^{n+m} - (\overline{b}-\overline{t})^{n+m}\right)$, it follows that

$$I_6(n,m-1) = \frac{(m-1)!n!}{(n+m)!}\left((\overline{b}-\underline{t})^{n+m} - (\overline{b}-\overline{t})^{n+m}\right) - \sum_{k=1}^{m-1} \frac{n!(m-1)!}{(m-k)!(n+k)!} S_1(k) \tag{24}$$

Substituting Equation (24) into Equation (20) derives Equation (9).

Derivation of equation (10)

For Case 3, Equation (6) is rewritten as

$$I = A\int_{\underline{t}}^{\overline{b}} (\overline{b}-x)^{n-1}\left(\int_{\underline{t}}^{x}(y-\underline{t})^{m-1}dy\right)dx - A\int_{\underline{t}}^{\underline{b}} (\overline{b}-x)^{n-1}\left(\int_{\underline{t}}^{x}(y-\underline{t})^{m-1}dy\right)dx \tag{25}$$

Note that the first term of the right hand side of Equation (25) is identical to Equation (12) in Case 1. Denote this term by I_7. Denote by I_8 the second term of the right hand side of the above equation.

$$\begin{aligned} I_8 &= -\frac{A}{m}\int_{\underline{t}}^{\underline{b}}(\overline{b}-x)^{n-1}(x-\underline{t})^m dx && (26)\\ &= \frac{A}{mn}(\overline{b}-\underline{b})^n(\underline{b}-\underline{t})^m - \frac{A}{n}\int_{\underline{t}}^{\underline{b}}(\overline{b}-x)^n(x-\underline{t})^{m-1}dx && (27) \end{aligned}$$

Put $I_9(n,m-1) = \int_{\underline{t}}^{\underline{b}}(\overline{b}-x)^n(x-\underline{t})^{m-1}dx$. By integrating by parts, we obtain

$$I_9(n,m-1) = \frac{m-1}{n+1}\int_{\underline{t}}^{\underline{b}}(\overline{b}-x)^{n+1}(x-\underline{t})^{m-2}dx - \frac{1}{n+1}(\overline{b}-\underline{b})^{n+1}(\underline{b}-\underline{t})^{m-1} \tag{28}$$

By almost the same calculation in I_6, we obtain the following.

$$I_9(n,m-1) = \frac{m!n!}{m(n+m)!}((\overline{b}-\underline{t})^{n+m} - (\overline{b}-\underline{b})^{n+m} - \sum_{k=1}^{m-1}\frac{n!(m-1)!}{(m-k)!(n+k)!}S_2(k) \tag{29}$$

Substituting (29) into (27) and adding (8) to (27) derives Equation (10).

Derivation of equation (11)

For Case 4, Equation (6) is rewritten as follows.

$$\begin{aligned} I &= A\int_{\underline{t}}^{\overline{t}}(\overline{b}-x)^{n-1}\left(\int_{\underline{t}}^{x}(y-\underline{t})^{m-1}dy\right)dx + A\int_{\underline{t}}^{\overline{b}}(\overline{b}-x)^{n-1}\left(\int_{\underline{t}}^{x}(y-\underline{t})^{m-1}dy\right)dx \\ &\quad -\frac{A}{mn}(\overline{b}-\underline{b})^n(\underline{b}-\underline{t})^m - \frac{A}{n}\int_{\underline{t}}^{\underline{b}}(\overline{b}-x)^n(x-\underline{t})^{m-1}dx && (30) \end{aligned}$$

Since the first term of Equation (30) is the same as Equation (17) for Case 2, and the second term is the same as I_8, we obtain (11) by similar calculation.

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