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E-Convexity of the Optimal Value Function in Parametric Nonlinear Programming

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Abstract Consider a general parametric optimization problem $P(\varepsilon)$ of the form $\min_x f(x,\varepsilon)$, s.t. $x \in R(\varepsilon)$. Convexity and generalized convexity properties of the optimal value function f^* and the solution set map S^* form an important part of the theoretical basis for sensitivity, stability, and parametric analysis in mathematical optimization. Fiacco and Kyparisis [1] systematically discussed the convexity and concavity of f^* for the above parametric program $P(\varepsilon)$ and its several special forms. In this paper, we extend these main results in [1] to the E-convexity of f^* by introducing E-convexity of set-valued maps.

Keywords Optimal value function; E-convex functions; E-quasiconvex functions; E-convex set-valued maps

1 Introduction

Let $\mathscr{R}^n$ denote the $n-$dimensional Euclidean space. We consider a general parametric optimization problem of the from

$$P(\varepsilon)\begin{cases} \min f(x,\varepsilon) \\ s.t.\ \ x\in R(\varepsilon), \end{cases}$$

where $f:\mathscr{R}^n\times\mathscr{R}^k\to\mathscr{R}^1$ and R is a set-valued map from $\mathscr{R}^k$ to $\mathscr{R}^n$, as well as several specializations of this problem. The optimal value function f^* of problem $P(\varepsilon)$ (sometimes called the perturbation function or the marginal function)is defined as

$$f^*=\begin{cases} \inf_x\{f(x,\varepsilon)|x\in R(\varepsilon)\}, & \text{if } R(\varepsilon)\neq\emptyset, \\ +\infty, & \text{if } R(\varepsilon)=\emptyset. \end{cases}$$

and the solution set-valued mappings S^* is defined by

$$S^*(\varepsilon)=\{x\in R(\varepsilon)|f(x,\varepsilon)=f^*(\varepsilon)\}.$$

We also consider the following several special programs of $P(\varepsilon)$:

$$P_1(\varepsilon)\begin{cases} \min_{x\in S} f(x,\varepsilon) \\ s.t.\ \ g_i(x,\varepsilon)\le 0, i=1,2,\cdots,m, \\ \qquad h_j(x,\varepsilon)=0, j=1,2,\cdots,p, \end{cases}$$

where $S \subset \mathscr{R}^n, g_i : \mathscr{R}^n \times \mathscr{R}^k \to \mathscr{R}^1, i = 1,2,\cdots,m, h_j : \mathscr{R}^n \times \mathscr{R}^k \to \mathscr{R}^1, j = 1,2,\cdots,p$, i.e., with R defined by

$$R(\varepsilon) = \{x \in S | g_i(x,\varepsilon) \le 0, i = 1,2,\cdots,m,\ h_j(x,\varepsilon) = 0, j = 1,2,\cdots,p\}.$$

$$P_2(\varepsilon) \begin{cases} \min\limits_{x \in S} f(x,\varepsilon) \\ s.t.\ \ g_i(x) \le \varepsilon_i, \quad i = 1,2,\cdots,m, \\ \qquad h_j(x) = \varepsilon_{m+j}, j = 1,2,\cdots,p, \end{cases}$$

where $S \subset \mathscr{R}^n, g_i : \mathscr{R}^n \times \mathscr{R}^k \to \mathscr{R}^1, i = 1,2,\cdots,m, h_j : \mathscr{R}^n \times \mathscr{R}^k \to \mathscr{R}^1, j = 1,2,\cdots,p$, i.e., with R defined by

$$R(\varepsilon) = \{x \in S | g_i(x) \le \varepsilon_i, i = 1,2,\cdots,m,\ h_j(x) = \varepsilon_{m+j}, j = 1,2,\cdots,p\}.$$

Convexity, concavity and other fundamental properties of the optimal value function f^* and the solution set-valued map S^*, such as continuity, differentiability, and so forth, form a theoretical basis for sensitivity, stability, and parametric analysis in nonlinear optimization. From the mid-1970s to the mid-1980s, the study of this area has been obtained intensively. Many papers had tired to unify these theories and methodologies, for instance [2-4]. Until 1986, Fiacco and Kyparisis[1] have systematically discussed the convexity and concavity of f^* for the above parametric program $P(\varepsilon)$ and its several special forms. Similarly, generalized convexity properties of the optimal value function f^* and the solution set map S^*, also play a role of theoretical basis for sensitivity, stability and parametric analysis in nonlinear programming. Zhang[5] discussed preinvexity and preincavity properites of f^*.

Recently, Youness [6] introduced a class of sets and a class of functions called E-convex sets and E-convex functions by relaxing the definitions of convex sets and convex functions, which has some important applications in various branches of mathematical sciences[7-9].

Motivated both by earlier research works and by the importance of the concepts of convexity and generalized convexity, we introduce the concepts of E-convex set-valued map and essentially E-convex set-valued map, and then develop some basic properties of E-convex and essentially E-convex set-valued maps. Based on theses new concepts, E-convexity properties of the optimal value function f^* for the parametric optimization problem $P(\varepsilon)$ and its several special forms are considered.

2 E-convexity of set-valued maps

In this section, we introduce two concepts of generalized convexity of set-valued maps. Troughtout this section, M is a nonempty subset in $\mathscr{R}^k$, and R is a set-valued map from M to $\mathscr{R}^n$.

Definition 2.1.([6]) A set M is said to be E-convex if there is a map $E : \mathscr{R}^k \to \mathscr{R}^k$ such that

$$(1-\lambda)E(x) + \lambda E(y) \in M,$$

for each $x, y \in M$ and $\lambda \in [0,1]$.

Lemma 2.1.([6]) If a set M is E-convex, then $E(M) \subset M$.

It is known from Lemma 2.1 that $E(M) \subseteq M$. Hence, for any set-valued map R, we have the following observations:

Observation(a) The set-valued map $R \circ E : M \to 2^{\mathscr{R}^n}$ defined by

$$(R \circ E)(x) = R(E(x)) \qquad \text{for all } x \in M$$

is well defined.

Observation(b) The Restriction $\tilde{R} : E(M) \to 2^{\mathscr{R}^n}$ of $R : M \to 2^{\mathscr{R}^n}$ to $E(M)$ defined by

$$\tilde{R}(\tilde{x}) = R(\tilde{x}) \qquad \text{for all } \tilde{x} \in E(M)$$

is well defined.

Definition 2.2.([1]) Let M be a convex set.

(1) The set-valued map R is called convex on M if, for any $\varepsilon_1, \varepsilon_2 \in M$ and $\lambda \in [0,1]$,

$$\lambda R(\varepsilon_1) + (1-\lambda)R(\varepsilon_2) \subset R(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2).$$

(2) The set-valued map R is called essentially convex on M if, for any $\varepsilon_1, \varepsilon_2 \in M, \varepsilon_1 \neq \varepsilon_2$ and $\lambda \in [0,1]$,

$$\lambda R(\varepsilon_1) + (1-\lambda)R(\varepsilon_2) \subset R(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2).$$

Based on the concept of convex set-valued maps and essentially convex set-valued maps, we introduce the concepts of E-convex set-valued maps and essentially E-convex set-valued maps.

Definition 2.3. (1) The set-valued map R is called E-convex on M if there is a map $E : \mathscr{R}^k \to \mathscr{R}^k$ such that M is an E-convex set and

$$\lambda(R \circ E)(\varepsilon_1) + (1-\lambda)(R \circ E)(\varepsilon_2) \subset R(\lambda E(\varepsilon_1) + (1-\lambda)E(\varepsilon_2)),$$

for any $\varepsilon_1, \varepsilon_2 \in M$ and $\lambda \in [0,1]$.

(2) The set-valued map R is called essentially E-convex on M if there is a map $E : \mathscr{R}^k \to \mathscr{R}^k$ such that M is an E-convex set and

$$\lambda(R \circ E)(\varepsilon_1) + (1-\lambda)(R \circ E)(\varepsilon_2) \subset R(\lambda E(\varepsilon_1) + (1-\lambda)E(\varepsilon_2)),$$

for any $\varepsilon_1, \varepsilon_2 \in M, E(\varepsilon_1) \neq E(\varepsilon_2)$ and $\lambda \in [0,1]$.

Remark 2.1. If R is convex (resp. essentially convex) on M, then R is E-convex (resp. essentially E-convex) on M.

Remark 2.2. If R is E-convex on M, then it is essentially E-convex on M. However, the converse is not true. See example 2.1.

Remark 2.3. If R is E-convex on M, then it is convex-valued with respect to E on M, i.e., $(R \circ E)(\varepsilon)$ at each $\varepsilon \in M$ is a convex set. However, An essentially convex set-valued map may not be convex-valued with respect to E at the boundary points of M, as shown below.

Example 2.1. Let $E : \mathscr{R}^2 \to \mathscr{R}^2$ be an identify map, $R : \mathscr{R}^2 \to R^1$ defined by

$$R(\varepsilon_1,\varepsilon_2) = \begin{cases} [0,1], & \text{if } \varepsilon_1^2+\varepsilon_2^2 < 1, \\ \{0\}\cup\{1\}, & \text{if } \varepsilon_1^2+\varepsilon_2^2 = 1, \\ \emptyset, & \text{if } \varepsilon_1^2+\varepsilon_2^2 > 1. \end{cases}$$

and

$$M = \{(\varepsilon_1,\varepsilon_2)|\varepsilon_1^2+\varepsilon_2^2 \le 1\}.$$

It is easy to check that R is essentially E-convex on M, but $(R\circ E)(\varepsilon_1,\varepsilon_2)$ is not convex if $\varepsilon_1^2+\varepsilon_2^2 = 1$.

From now on, let E be a map from $\mathscr{R}^k$ to $\mathscr{R}^k$ and M be a nonempty E-convex set.

Proposition 2.1. Let R be E-convex (resp. essentially E-convex) on M. Then the restriction, say $\bar{R} : C \to 2^{\mathscr{R}^n}$, of R to any nonempty convex subset C of $E(M)$ is convex (resp. essentially convex) on C.

Proof. Let $C \subset E(M)$ be convex, and let $\bar{\varepsilon_1},\bar{\varepsilon_2} \in C$ ($\bar{\varepsilon_1}$ and $\bar{\varepsilon_2}$ may not be distinct). Then there exist $\varepsilon_1,\varepsilon_2 \in M$ such that $\bar{\varepsilon_1} = E(\varepsilon_1)$ and $\bar{\varepsilon_2} = E(\varepsilon_2)$. Since $\lambda\bar{\varepsilon_1}+(1-\lambda)\bar{\varepsilon_2} \in C$, it follows from the E-convexity of R that

$$\begin{aligned} \lambda\bar{R}(\bar{\varepsilon_1})+(1-\lambda)\bar{R}(\bar{\varepsilon_2}) &= \lambda\bar{R}(E(\varepsilon_1))+(1-\lambda)\bar{R}(E(\varepsilon_2)) \\ &= \lambda(R\circ E)(\varepsilon_1)+(1-\lambda)(R\circ E)(\varepsilon_2) \\ &\subset R(\lambda E(\varepsilon_1)+(1-\lambda)E(\varepsilon_2)) \\ &= \bar{R}(\lambda\bar{\varepsilon_1}+(1-\lambda)\bar{\varepsilon_2}) \end{aligned}$$

for all $\lambda \in [0,1]$. Hence, $\bar{R}$ is convex on C.

Corollary 2.1. Let R be E-convex (resp. essentially E-convex) on M. If $E(M) \subset M$ is a convex set, then the restriction $\tilde{R} : E(M) \to 2^{\mathscr{R}^n}$ of R to $E(M)$ is convex (resp. essentially convex) on $E(M)$.

Proposition 2.2. Let $E(M) \subset M$ be a convex set . If the restriction $\tilde{R} : E(M) \to 2^{\mathscr{R}^n}$ of R to $E(M)$ is convex (resp. essentially convex) on $E(M)$, then R is E-convex (resp. essentially E-convex) on M.

Proof. Let $\varepsilon_1,\varepsilon_2 \in M$. Then $E(\varepsilon_1),E(\varepsilon_2) \in E(M)$, and by the convexity of $E(M)$, we can obtain $\lambda E(\varepsilon_1)+(1-\lambda)(\varepsilon_2) \in E(M)$ for all $\lambda \in [0,1]$. Since $\tilde{R}$ is convex on $E(M)$, we have

$$\begin{aligned} \lambda(R\circ E)(\varepsilon_1)+(1-\lambda)(R\circ E)(\varepsilon_2) &= \lambda R(E(\varepsilon_1))+(1-\lambda)R(E(\varepsilon_2)) \\ &= \lambda\tilde{R}(E(\varepsilon_1))+(1-\lambda)\tilde{R}(E(\varepsilon_2)) \\ &\subset \tilde{R}(\lambda E(\varepsilon_1)+(1-\lambda)E(\varepsilon_2)) \\ &= R(\lambda E(\varepsilon_1)+(1-\lambda)E(\varepsilon_2)), \end{aligned}$$

which shows R is E-convex on M.

Corollary 2.2. Suppose that $E(M)$ be convex . Then R is E-convex (resp. essentially E-convex) on M if and only if its restriction $\tilde{R} : E(M) \to 2^{\mathscr{R}^n}$ is convex (resp. essentially convex) on $E(M)$.

Let the map $I\times E : \mathscr{R}^n\times\mathscr{R}^k \to \mathscr{R}^n\times\mathscr{R}^k$ be

$$(I\times E)(x,\varepsilon) = (x,E(\varepsilon)), \qquad \text{for any}(x,\varepsilon) \in \mathscr{R}^n\times\mathscr{R}^k.$$

Denote

$$G(R) = \{(x, \varepsilon) | x \in R(\varepsilon), \varepsilon \in M\}.$$

It is easy to show that $G(R)$ is $I \times E-$convex, if and only if

$$(\lambda x_1 + (1-\lambda)x_2, \lambda E(\varepsilon_1) + (1-\lambda)E(\varepsilon_2)) \in G(R)$$

for each $(x_1, \varepsilon_1), (x_2, \varepsilon_2) \in G(R)$ and $\lambda \in [0,1]$.

Proposition 2.3. Suppose R is E-convex on M. If $R(\varepsilon) \subset (R \circ E)(\varepsilon)$ for each $\varepsilon \in M$, then $G(R)$ is $I \times E-$convex.

Proof. Let $(x_1, \varepsilon_1), (x_2, \varepsilon_2) \in G(R)$ and $\lambda \in [0,1]$. Then, $x_1 \in R(\varepsilon_1)$,$x_2 \in \varepsilon R(\varepsilon_2)$. By the assumption that $R(\varepsilon) \subset (R \circ E)(\varepsilon)$, we obtain

$$x_1 \in (R \circ E)(\varepsilon_1), \;\; x_2 \in (R \circ E)(\varepsilon_2). \tag{2.1}$$

Since R is E-convex on M and (2.1), we get

$$\lambda x_1 + (1-\lambda)x_2 \in R(\lambda E(\varepsilon_1) + (1-\lambda)E(\varepsilon_2)), \tag{2.2}$$

which means that $(\lambda x_1 + (1-\lambda)x_2, \lambda E(\varepsilon_1) + (1-\lambda)E(\varepsilon_2)) \in G(R)$. Therefore, $G(R)$ is $I \times E-$convex.

Proposition 2.4. Suppose $G(R)$ is $I \times E-$convex. If $(R \circ E)(\varepsilon) \subset R(\varepsilon)$ for each $\varepsilon \in M$, then R is E-convex on M.

Proof. Let $\varepsilon_1, \varepsilon_2 \in M$ and $\lambda \in [0,1]$. Take arbitrary points $x_1 \in (R \circ E)(\varepsilon_1)$,$x_2 \in (R \circ E)(\varepsilon_2)$. Then, it follows from $(R \circ E)(\varepsilon) \subset R(\varepsilon)$ for each $\varepsilon \in M$

$$x_1 \in R(\varepsilon_1), \;\; x_2 \in R(\varepsilon_2). \tag{2.3}$$

That is,

$$(x_1, \varepsilon_1), (x_2, \varepsilon_2) \in G(R). \tag{2.4}$$

Since $G(R)$ is $I \times E-$convex and (2.4), we get

$$(\lambda x_1 + (1-\lambda)x_2, \lambda E(\varepsilon_1) + (1-\lambda)E(\varepsilon_2)) \in G(R). \tag{2.5}$$

That is,

$$\lambda x_1 + (1-\lambda)x_2 \in R(\lambda E(\varepsilon_1) + (1-\lambda)E(\varepsilon_2)),$$

which shows that R is E-convex on M.

3 E-convexity of the optimal value function

In this section, we give the main results.

Definition 3.1.([6]) A function $g : \mathscr{R}^k \to \mathscr{R}^1$ is said to be E-convex on a set $M \subset \mathscr{R}^k$ if there is a map $E : \mathscr{R}^k \to \mathscr{R}^k$ such that M is an E-convex set and

$$g(\lambda E(\varepsilon_1) + (1-\lambda)E(\varepsilon_2)) \leq \lambda g(E(\varepsilon_1)) + (1-\lambda)g(E(\varepsilon_2)),$$

for each $\varepsilon_1, \varepsilon_2 \in M$ and $\lambda \in [0,1]$.

It is easy to show that $f : \mathscr{R}^n \times \mathscr{R}^k \to \mathscr{R}^1$ is $(I \times E)-$convex on $\mathscr{R}^n \times M$, if and only if

$$f(\lambda x_1 + (1-\lambda)x_2, \lambda E(\varepsilon_1) + (1-\lambda)E(\varepsilon_2)) \leq \lambda f(x_1, E(\varepsilon_1)) + (1-\lambda) f(x_2, E(\varepsilon_2))$$

for each $(x_1, \varepsilon_1), (x_2, \varepsilon_2) \in \mathscr{R}^n \times M$ and $\lambda \in [0,1]$.

Theorem 3.1. Consider the general parametric optimization problem $P(\varepsilon)$. if f is $(I \times E)-$convex on the set $\{(x, \varepsilon) | x \in R(E(\varepsilon)), \varepsilon \in M\}$, R is essentially E-convex on M, and M is E-convex, then f^* is E-convex on M.

Proof. Let $\varepsilon_1, \varepsilon_2 \in M, \varepsilon_1 \neq \varepsilon_2$, and $\lambda \in [0,1]$. Then, by the $(I \times E)-$convexity of f and essential E-convexity of R, we obtain

$$\begin{aligned}
& f^*(\lambda E(\varepsilon_1) + (1-\lambda)E(\varepsilon_2)) \\
= & \inf_{x \in R(\lambda E(\varepsilon_1) + (1-\lambda)E(\varepsilon_2))} f(x, \lambda E(\varepsilon_1) + (1-\lambda)E(\varepsilon_2)) \\
\leq & \inf_{x_1 \in (R \circ E)(\varepsilon_1), x_2 \in (R \circ E)(\varepsilon_2)} f(\lambda x_1 + (1-\lambda)x_2, \lambda E(\varepsilon_1) + (1-\lambda)E(\varepsilon_2)) \\
\leq & \inf_{x_1 \in (R \circ E)(\varepsilon_1), x_2 \in (R \circ E)(\varepsilon_2)} [\lambda f(x_1, E(\varepsilon_1)) + (1-\lambda) f(x_2, E(\varepsilon_2))] \\
= & \lambda \inf_{x_1 \in (R \circ E)(\varepsilon_1)} f(x_1, E(\varepsilon_1)) + (1-\lambda) \inf_{x_2 \in (R \circ E)(\varepsilon_2)} f(x_2, E(\varepsilon_2)) \\
= & \lambda f^*(E(\varepsilon_1)) + (1-\lambda) f^*(E(\varepsilon_2)),
\end{aligned}$$

i.e., f^* is E-convex on M.

Definition 3.2.([10]) A function $g : \mathscr{R}^k \to \mathscr{R}^1$ is said to be E-quasiconvex on a set $M \subset \mathscr{R}^k$ if there is a map $E : \mathscr{R}^k \to \mathscr{R}^k$ such that M is an E-convex set and

$$g(\lambda E(\varepsilon_1) + (1-\lambda)E(\varepsilon_2)) \leq \max\{g(E(\varepsilon_1)), g(E(\varepsilon_2))\},$$

for each $\varepsilon_1, \varepsilon_2 \in M$ and $\lambda \in [0,1]$.

The functions g is said to E-quasiconcave, if $-g$ is E-quasiconvex; g is said to E-quasimonotonic, if g both is E-quasiconvex and E-quasiconcave.

Theorem 3.2. Consider the parametric problem $P_1(\varepsilon)$. if g_i are $(I \times E)-$quasiconvex on $S \times M$, h_j are $(I \times E)-$quasimonotonic on $S \times M$, S is convex and M is E-convex, then R, given by

$$R(\varepsilon) = \{x \in S | g_i(x, \varepsilon) \leq 0, i = 1, 2, \cdots, m, \ h_j(x, \varepsilon) = 0, j = 1, 2, \cdots, p\},$$

is E-convex on M.

Proof. Let $\varepsilon_1, \varepsilon_2 \in M$ and take arbitrary points $x_1 \in (R \circ E)(\varepsilon_1), x_2 \in (R \circ E)(\varepsilon_2)$. Then, $x_1, x_2 \in S$,

$$g_i(x_1, E(\varepsilon_1)) \leq 0, g_i(x_2, (E\varepsilon_2)) \leq 0, i = 1, 2, \cdots, m \tag{3.1}$$

and

$$h_j(x_1, E(\varepsilon_1)) = 0, h_j(x_2, (E(\varepsilon_2)) = 0, j = 1, 2, \cdots, p. \tag{3.2}$$

Since S is convex and M is E-convex, we have

$$\lambda x_1 + (1-\lambda)x_2 \in S \text{ and } \lambda E(\varepsilon_1) + (1-\lambda)E(\varepsilon_2) \in M \quad \text{for any } \lambda \in [0,1]. \tag{3.3}$$

By $(I \times E)-$quasiconvexity of g_i on $S \times M$ and (3.1), we obtain

$$\begin{aligned} g_i(\lambda x_1 + (1-\lambda)x_2, \lambda E(\varepsilon_1) + (1-\lambda)E(\varepsilon_2)) &\leq \max\{g_i(x_1, E(\varepsilon_1)), g_i(x_2, E(\varepsilon_2))\} \\ &\leq 0. \end{aligned} \tag{3.4}$$

Similarly, by$(I \times E)-$quasimonotonic of h_j on $S \times M$ and (3.2), we can get

$$h_j(\lambda x_1 + (1-\lambda)x_2, \lambda E(\varepsilon_1) + (1-\lambda)E(\varepsilon_2)) = 0. \tag{3.5}$$

Therefore, by (3.3-3.5), we obtain

$$\lambda x_1 + (1-\lambda)x_2 \in R(\lambda E(\varepsilon_1) + (1-\lambda)E(\varepsilon_2)),$$

which means that $\lambda(R \circ E)(\varepsilon_1) + (1-\lambda)(R \circ E)(\varepsilon_2) \subset R(\lambda E(\varepsilon_1) + (1-\lambda)E(\varepsilon_2))$, i.e., R is E-convex on M.

The following result is now immediate.

Corollary 3.1. Consider the parametric problem $P_1(\varepsilon)$. if f is $(I \times E)-$convex on the set $\{(x, \varepsilon) | x \in R(E(\varepsilon)), \varepsilon \in M\}$, g_i are $(I \times E)-$quasiconvex on $S \times M$, h_j are $(I \times E)-$quasimonotonic on $S \times M$, S is convex and M is E-convex, then f^* is E-convex on M.

Proof. This follows directly from Theorems 3.1 and Theorems 3.2.

The next result follows directly from Theorems 3.2.

Theorem 3.3. Consider the parametric problem $P_2(\varepsilon)$. if g_i are $(I \times E)-$quasiconvex on $S \times M$, h_j are $(I \times E)-$quasimonotonic on $S \times M$, S is convex and M is E-convex, then R, given by

$$R(\varepsilon) = \{x \in S | g_i(x) \leq \varepsilon_i, i = 1, 2, \cdots, m,\ h_j(x) = \varepsilon_{m+j}, j = 1, 2, \cdots, p\},$$

is E-convex on M.

Corollary 3.2. Consider the parametric problem $P_2(\varepsilon)$. if f is $(I \times E)-$convex on the set $\{(x, \varepsilon) | x \in R(E(\varepsilon)), \varepsilon \in M\}$, g_i are $(I \times E)-$quasiconvex on $S \times M$, h_j are $(I \times E)-$quasimonotonic on $S \times M$, S is convex and M is E-convex, then f^* is E-convex on M.

Proof. This follows directly from Theorems 3.1 and Theorems3.3.

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