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Tractability and Intractability of Problems on Unit Disk Graphs Parameterized by Domain Area

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Abstract This paper treats unit disk graphs whose vertices are located in a square-shaped region with fixed area α, and considers parametrized problems on this model. It shows that "fixed area" is not a trivial restriction by proving that the maximum independent set problem and the minimum dominating set problem are both W[1]-complete for unit disk graphs parameterized by area. On the other hand, it shows an algorithm that solves the Hamiltonian circuit problem in $O(m+p^2c^p)$ time, where m is the number of edges, $p=2\alpha+o(\alpha)$, and c is a constant number, i.e., this problem is FPT for the parameter α. It also shows an algorithm that solves the k-coloring problem in $O(k^{kp})$ time, i.e., this problem is also FPT for the pair of parameters k and α.

Keywords Unit disk graphs, Hamiltonian circuit, FPT, W[1]-hard, Independent set, Dominating set, Coloring, Ad-hoc networks

1 Introduction

Many combinatorial problems remain NP-hard even if they are restricted on unit disk graphs. However such instances used for the proof of the NP-hardness occupy (certainly polynomial, but) wide areas. Considering applications, e.g. ad-hoc networks [14], the areas are normally not so wide. So we consider unit disk graphs on a domain parameterized by area $\alpha > 0$: all centers of disks are on a square with area α, i.e., the length of a side of the square is $\sqrt{\alpha}$.

We consider the following four problems on this model: HAMILTONIAN CIRCUIT, k-COLORING, MAXIMUM INDEPENDENT SET and MINIMUM DOMINATING SET, which are all NP-hard even on unit disk graphs. (For HAMILTONIAN CIRCUIT see [6]. For the other problems see [4].) We show that HAMILTONIAN CIRCUIT can be solved in $O(m+p^2c^p)$ time, where m is the number of edges, $p=2\alpha+2\sqrt{2\alpha}+1=2\alpha+o(\alpha)$, and $c=18^{18}$. We next show that k-COLORING can be solved in $O(k^{k\cdot p})$ time. That is, they are both in FPT (k-COLORING has two parameters α and k). On the contrary we prove that MAXIMUM INDEPENDENT SET and MINIMUM DOMINATING SET are both W[1]-complete. From the latter results, we can see that *fixed area is not a trivial restriction*. Our results are summarized in Table 1.

We briefly survey related work. Clark, Colbourn and Johnson [4] showed that MAXIMUM INDEPENDENT SET, MINIMUM VERTEX COVER are NP-complete even for unit

Table 1: Summary of classical and parameterized complexity on problems for unit disk graphs

Problem	classical complexity	parameterized by area (our results)
HAMILTONIAN CIRCUIT	NP-complete [6]	FPT
k-COLORING	NP-complete [4]	FPT
MAXIMUM INDEPENDENT SET	NP-complete [4]	W[1]-complete
MINIMUM DOMINATING SET	NP-complete [10]	W[1]-complete

disk graphs. They showed that k-COLORING is NP-complete for unit disk graphs even k is fixed by 3. Itai, Papadimitriou and Szwarcfiter [6] showed that HAMILTONIAN CIRCUIT is NP-complete for grid graphs. Since the class of grid graphs is a subclass of unit disk graphs, HAMILTONIAN CIRCUIT is also NP-complete for unit disk graphs. Some PTASs which solve MAXIMUM INDEPENDENT SET for unit disk graphs have been shown [5, 12]. MINIMUM CONNECTED DOMINATING SET has a direct application to virtual backbone, which is useful for routing in ad-hoc networks, and hence many approximation algorithms for solving the problem on unit disk graphs have been studied [2, 7, 11]. Exact algorithms for unit disk graphs have been also studied. Alber [1] showed that k-INDEPENDENT SET on λ-precision disk graph with bounded radius ratio is FPT with running time $2^{O(\sqrt{k})} + n^{O(1)}$, using geometric separation theorem. For a recent survey on optimization problems on unit disk graphs, see [3].

2 General Property of Unit Disk Graphs of Fixed Area

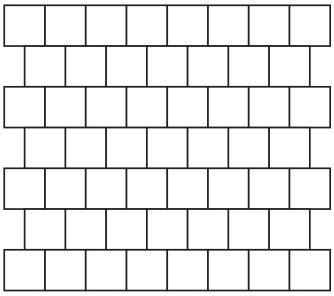
A graph $G = (V, E)$ with n vertices and m edges is a unit disk graph if its vertices can be put in one-to-one correspondence with unit circles in the plane in such a way that two vertices are joined by an edge if and only if the corresponding circles intersect (or they are tangent). We assume $n = O(m)$. In this paper the correspondences of vertices and circles is given, i.e., a unit disk graph is represented by a set of unit circles in the plane. Moreover, we focus on unit disk graphs whose vertices (the center of unit circles) are placed in a square which is denoted by $\mathscr{S}$, with area α. For considering this model, the idea of clique partition are useful.

Definition 2.1 (Clique Partition)**.**
Let $G = (V, E)$ be a graph. For an integer p, a partition $\mathscr{Q} = \{Q_1, Q_2, \cdots Q_p\}$ ($Q_i \cap Q_j = \emptyset$ for $1 \le i < j \le p$, and $\bigcup_{i=1}^{p} Q_i = V$) of V is a clique partition *if each Q_i is a clique.*

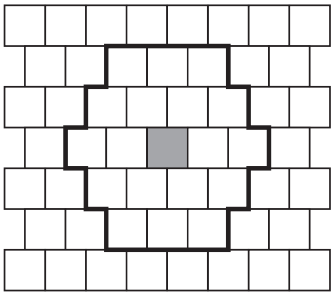
A unit disk graph on $\mathscr{S}$ has a clique partition with small size:

Lemma 2.2.
A unit disk graph G on $\mathscr{S}$ has a clique-partition with the size at most $p = 2\alpha + 2\sqrt{2\alpha} + 1 = 2\alpha + o(\alpha)$.

Proof. $\mathscr{S}$ can be covered by at most $(\sqrt{2\alpha} + 1)^2 = 2\alpha + 2\sqrt{2\alpha} + 1$ tiles of square with side $1/\sqrt{2}$. Each tile induces a clique, i.e., the covering gives a clique partition. □



(a)



(b)

Figure 1: (a) Example of the tiling. (b) Any vertex in the central tile is not adjacent to vertices outside of the bold boundary.

Someone may cursorily thinks that the clique partition may directly make all problems trivial. But it is not true. Certainly there are some W[1]-hard problems as shown in section 4. Moreover even for HAMILTONIAN CIRCUIT, the clique partition is not sufficient to obtain an FPT algorithm. (This is shown in section 3).

From now, we consider the clique partition $\mathcal{Q} = \{Q_1, \cdots, Q_p\}$ $(p = 2\alpha + 2\sqrt{2\alpha} + 1)$ defined in Lemma 2.2. We use honeycomb-like arrangement of squares as shown in Fig. 1 (a). By using such an arrangement, if $v_i \in Q_i$ is adjacent to $v_j \in Q_j$ $(i \neq j)$ then v_j lies in one of $\Delta(= 18)$ tiles close to Q_i as shown in Fig. 1 (b). That is, Δ means an upper bound of the number of "adjacent" cliques for any cliques $Q_i \in \mathcal{Q}$.

We define some signs in this paper: For vertex subsets $X, Y \subseteq W$, $E(X,Y)$ denotes the set of edges between X and Y, i.e., $E(X,Y) = \{xy \in E | x \in X, y \in Y\}$. $E(X) = E(X,X) = \{xy \in E | x, y \in X\}$. Two edges are *independent* if they have no common end vertices.

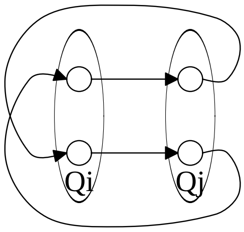
3 Hamiltonian Circuit

HAMILTONIAN CIRCUIT is a problem of deciding whether or not a given graph has a Hamiltonian circuit, which is a circuit containing every vertex in the graph just one time. In this section we show HAMILTONIAN CIRCUIT on unit disk graphs is FPT w.r.t. the parameter α. We mainly present two lemmas for helping us. One is for decreasing the number of Hamiltonian circuit that we should consider (Lemma 3.2), and the other is for limiting edges to be examined (Lemma 3.3).

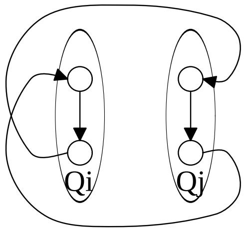
Definition 3.1.
A canonical *Hamiltonian circuit C is a Hamiltonian circuit such that for arbitrary $i, j \in \{1, \cdots, p\}, i \neq j$, C contains at most two edges in $E(Q_i, Q_j)$.*

Lemma 3.2.
G has a Hamiltonian Circuit if and only if G also has a canonical Hamiltonian circuit.

Proof. "If" part is trivial, and hence we show "only if" part. We assume that G has a Hamiltonian circuit C_0. We also assume there is a pair $Q_i, Q_j \in \mathcal{Q}$ such that $C_0 \cap E(Q_i, Q_j)$



(a) C_0



(b) C_0'

Figure 2: Example of constructing C_0' (b) by replacing edges in C_0 (a)

includes more than two edges. It is enough to show that there is another Hamiltonian circuit C_0' such that C_0' has at most two edges in $E(Q_i,Q_j)$ and both C_0 and C_0' have the same edges in $E \setminus (E(Q_i) \cup E(Q_j) \cup E(Q_i,Q_j))$. (Because we can apply this property for every part of $Q_{i'}, Q_{j'} \in \mathscr{Q}$ having more than two edges in $C_0 \cap E(Q_{i'},Q_{j'})$ one by one.) We direct all edges in C_0 in the same direction along the circuit. Let m_{ij} (resp., m_{ji}) be the number of edges in $C_0 \cap E(Q_i,Q_j)$ which is directed from Q_i to Q_j (resp., Q_j to Q_i). Since we assumed $m_{ij}+m_{ji} > 2$, we can also assume $m_{ij} \geq 2$ without loss of generality. Then we can decrease m_{ij} by two by changing a part of the circuit as shown in Fig. 2. By applying this procedure repeatedly, m_{ij} becomes one or zero, i.e., $m_{ij}+m_{ji}$ is at most two. □

We consider a graph $H_0 = (U,F_0)$ obtained from G by contracting each Q_i to a vertex u_i. Let H be a multigraph obtained from H_0 by doubling an edge u_iu_j if $|E(Q_i,Q_j)| \geq 2$. For any vertex $u_i \in U$ in H, the number of vertices adjacent to u_i is at most $\Delta = 18$.

A *spanning circuit*, or an *s-circuit*, in short is a circuit that passes all vertices of H. By Lemma 3.2, if G has a Hamiltonian circuit, G has a canonical Hamiltonian circuit C. The graph obtained from C by contracting Q_i to u_i is an s-circuit. Since the maximum length of an s-circuit in H is Δp, and hence the number of possible s-circuits in H is at most $\Delta^{\Delta p}$. From this fact, we can decide the existence of a Hamiltonian circuit in G by the following way: Enumerate all possible s-circuits D, and check whether a canonical Hamiltonian circuit corresponding to D exists or not. We show an algorithm for deciding whether G has a Hamiltonian circuit corresponding to a given s-circuit D in time $O(p^2)$ with a preprocessing requiring $O(m)$ computation time. Note that if we naively enumerate all possible combinations of edges for each pair of vertices of H, then the computation time becomes $\Omega(n^p)$, which is not FPT. We show that choosing at most $4(2\Delta-1)^2$ edges between each pair Q_i,Q_j as candidates is sufficient to examine the existence of such a canonical Hamiltonian circuit.

Lemma 3.3.
We assume that G has a Hamiltonian circuit C corresponding to an s-circuit in H. Then for arbitrary $i,j \in \{1,2,\cdots,p\}$ $(i \neq j)$, there exists an edge subset $E'_{ij} \subseteq E(Q_i,Q_j)$, ($|E'_{ij}| \leq 4(2\Delta-1)^2$), such that G has a canonical Hamiltonian circuit C' satisfying the following two conditions:

1. *C and C' consists of the same edges except in $E(Q_i) \cup E(Q_j) \cup E(Q_i, Q_j)$, i.e., $C \setminus (E(Q_i) \cup E(Q_j) \cup E(Q_i, Q_j)) = C' \setminus (E(Q_i) \cup E(Q_j) \cup E(Q_i, Q_j))$,*
2. *$C' \cap E(Q_i, Q_j) \subseteq E'_{ij}$.*

Proof. In this proof E'_{ij} is expressed as E' for notational simplicity. We show an algorithm for constructing E'. First, we set $E' = \emptyset$. We compute an arbitrary maximal matching M of a bipartite graph $B = (Q_i \cup Q_j, E(Q_i, Q_j))$ whose parts are Q_i and Q_j, (If there is a matching with size at least two then we choose $|M|$ to be $|M| \geq 2$.) There are two cases :

1. Case: $|M| \geq 4\Delta - 2$. Choose arbitrary $4\Delta - 2$ edges from M and add them to E'.
2. Otherwise. Add all edges in M to E'. For each vertex v to which an edge in M is incident, choose at most $2\Delta - 1$ arbitrary edges incident to v as much as possible from $E(Q_i, Q_j)$. Add these edges to E'.

For resulting E', $|E'| \leq (4\Delta - 2) \cdot 2 \cdot (2\Delta - 1) = 4(2\Delta - 1)^2$. We show that G has a canonical Hamiltonian Circuit C' which consists of edges in E'.

Assume $C \cap E(Q_i, Q_j) \neq \emptyset$. $C \cap E(Q_i, Q_j)$ consists of one or two edges, since it is a canonical Hamiltonian circuit. If $C \cap E(Q_i, Q_j) \subseteq E'$, then we must do nothing. (The desired canonical Hamiltonian circuit C' is already obtained.) Then assume that there is at least one edge e_0 such that $e_0 \in C \cap E(Q_i, Q_j)$ and $e_0 \notin E'$. In $C \setminus E(Q_i) \cup E(Q_j) \cup E(Q_i, Q_j)$, there exist at most $4\Delta - 4$ edges incident to vertices in $Q_i \cup Q_j$. This is because C has at most 2Δ edges vv_i with $v \notin Q_i$ and $v_i \in Q_i$ for an arbitrary Q_i. Let $E'(u) \subseteq E'$ be an edge set incident to u.

1. Case: $|M| = 1$. From the algorithm for constructing E', $|M| = 1$ means that there is no matching with size larger than one in B. Hence there is a vertex, say v, in $Q_i \cup Q_j$ such that all edges in $E(Q_i, Q_j)$ are incident to v. That is, $E' = E'(v)$. Assume $v \in Q_i$ w.l.o.g. Clearly $C \cap E(Q_i, Q_j)$ consists of one edge $e_0 = vw$. From the assumption of $e_0 \notin E'$ and the algorithm for constructing E', it follows that $|E'(v)| = 2\Delta$. Hence there is at least one edge $e = vu \in E'(v)$ such that $u \in Q_j$ is not incident to any edges in C. Then we get C' from C by replacing $e_0 = vw$ with $e = vu$ and uw.
2. Case: $2 \leq |M| \leq 4\Delta - 3$. Note that $M \subseteq E'$ in this case. From that M is a maximal matching, it follows that there is an edge $e' \in M$ which is adjacent with e_0, i.e., there is a vertex v incident to both e_0 and e'. Assume w.l.o.g. $v \in Q_i$. Let $e_0 = vw$ and $e' = vw'$. From the assumption $e_0 = vw \notin M \subseteq E'$. From the assumption that $e_0 = vw \notin E'$, it follows that $E'(v)$ is a proper subset of $E(\{v\}, Q_j)$. and $|E'(v)| = 2\Delta$. Thus there at least two edges $e'' = vw'', e''' = vw''' \in E'(v)$ such that w'' and w''' are not incident to any edge in $C \setminus (E(Q_i) \cup E(Q_j) \cup E(Q_i, Q_j))$. We can easily replace $e_0 = vw$ with $e'' = vw''$ and $w''w$ (or $e''' = vw'''$ and $w'''w$).
3. Case: $|M| \geq 4\Delta - 2$. Note that $|M \cap E'| = 4\Delta - 2$ in this case. Then there exist at least two edges $e', e'' \in M \cap E'$ such that e' and e'' are independent of all edges in $C \setminus (E(Q_i) \cup E(Q_j) \cup E(Q_i, Q_j))$. Clearly we can use e', e'' instead of (at most two) edges in $C \cap E(Q_i, Q_j)$ as the former cases.

□

Now we have the algorithm: Make a subgraph G' of G by selecting $E'_{ij} \subseteq E(Q_i, Q_j)$ consisting of at most $4(2\Delta - 1)^2$ edges for each pair of Q_i and Q_j. That is, $G' = (V, \bigcup_{i,j} E'_{ij})$.

There are at most $\Delta^{\Delta p}$ possible s-circuits in H. For each s-circuit D, we determine whether or not G' has a canonical Hamiltonian circuit corresponding to D by enumerating all possible candidates of edges from every E'_{ij}. The detailed discussion is as follows.

Theorem 3.4.
HAMILTONIAN CIRCUIT *is solvable in time* $O(m + c^p p^2)$ *for unit disk graphs whose vertices are placed in* $\mathscr{S}$*, which is a asuare with area* α*, where* $p = 2\alpha + 2\sqrt{2\alpha} + 1$ *and* $c = 18^{18}$.

Proof. Constructing G' can be done in $O(m)$ time. The number of vertices and edges of H is at most p and Δp, respectively. There are at most $\Delta^{\Delta p}$ possible s-circuits in H.

G' has at most $\Delta p/2$ pairs (Q_i, Q_j) having an edge between them. For each of such pairs (Q_i, Q_j), we choose in order at most two edges from E'_{ij} for candidates for a canonical Hamiltonian circuit corresponding D. The number of possible such candidates of whole G' is at most $(4(2\Delta - 1)^2)^2 \times \frac{\Delta p}{2} = 8(2\Delta - 1)^4 \Delta p$.

For each pair of D and E', determining whether or not E' leads a canonical Hamiltonian circuit corresponding to D is easily determined in time $O(p)$. Therefore the total computation time is $O(m + \Delta^{\Delta p} \cdot 8(2\Delta - 1)^4 \Delta p \cdot p) = O(m + p^2 \Delta^{\Delta p}) = O(m + p^2 c^p)$. □

4 Other Problems

4.1 k-Coloring

Let G be a unit disk graph on $\mathscr{S}$. In this section, we show an FPT algorithms for k-COLORING on unit disk graphs on $\mathscr{S}$. G is called *k-colorable* if there is a function $\Gamma : V \to \{1, 2, \cdots, k\}$ such that $\Gamma(v) \neq \Gamma(w)$ for all $vw \in E$. k-COLORING is a problem for determining where or not a given graph G is k-colorable.

Theorem 4.1.
k-Coloring for unit disk graphs on $\mathscr{S}$ *is solvable in time* $O(k^{k \cdot p})$.

Proof. If G is k-colorable, G does not include a clique with size larger than k. G has a clique partition $\mathscr{Q} = \{Q_1, Q_2, \cdots, Q_p\}$. Hence we can reject G if it has more than kp vertices. For a graph with at most kp vertices, k-coloring can be solved in $O(k^{k \cdot p})$ time by testing all possible assignments. □

4.2 Maximum Independent Set

The previous results are on tractability. On the other hand, in this and the next subsections, we show intractable problems for fixed area unit disk graphs. For a graph $G = (V, E)$, a vertex subset $W \subseteq V$ is an *independent set* if for every $v, w \in W$ there is no edge between them, i.e., $vw \notin E$. The size of an independent set W is $|W|$. MAXIMUM INDEPENDENT SET is a problem of deciding whether or not a given graph has an independent set with size of a given integer k. We consider this problem for unit disk graphs parameterized by area α. The major parameterized approach for MAXIMUM INDEPENDENT SET is to use the parameter k. We call this problem k-INDEPENDENT SET. We find a close relation between these two parameterized problems.

Lemma 4.2.
There exists an FPT algorithm for MAXIMUM INDEPENDENT SET *for unit disk graphs parameterized by* α *if and only if there exists an FPT algorithm for* k-INDEPENDENT SET *for unit disk graphs.*

Proof. "If" part is easy. For a unit disk graph G on the square $\mathscr{S}$, the size of independent set is at most p. Then by solving k-INDEPENDENT SET for all $k = 1, 2, \cdots, p$ MAXIMUM INDEPENDENT SET for G can be solved. Thus "if" part follows.

To show "only if" part, we assume that there exists an FPT algorithm $\Psi(G, \alpha)$ which solves MAXIMUM INDEPENDENT SET for unit disk graphs on $\mathscr{S}$ in time $f'(\alpha)\text{poly}(|G|)$, where α is the area of $\mathscr{S}$. Given an arbitrary unit disk graph G_0 and an integer k, we show an algorithm for computing whether or not G contains independent set of size k. Let $G_1, \cdots, G_{k'}$ be connected components of G. If $k' \geq k$ then G has an independent set of size k clearly. Hence we assume $k' < k$. By using a greedy algorithm, we can easily obtain a maximal (not maximum) independent set S_i for every G_i. We assume $|S_1| + \cdots + |S_{k'}| < k$, otherwise G has an independent set of size k. For any G_i, from that it is connected, it follows that the diameter of G_i is at most $3|S_i| - 1 < 3k - 1$. From this, we can observe that all vertices in each G_i are in a square of area $(2(3k-1))^2 < 36k^2$. Applying $\Psi(G_i, 36k^2)$ for each $i = \{1, 2, \cdots, k'\}$, we can get a maximum independent set of G_i in time at most $f(36k^2)\text{poly}(|G_i|)$. The total computing time is $T = O\left(km + kf(36k^2) \cdot \text{poly}(n)\right)$. □

Marx [8, 9] showed that k-INDEPENDENT SET is W[1]-complete for unit disk graphs. Combine Lemma 4.2 with [8, 9], we get the following theorem.

Theorem 4.3.
MAXIMUM INDEPENDENT SET *on unit disk graphs parameterized by area* α *is W[1]-complete.* □

4.3 Minimum Dominating Set

For MINIMUM DOMINATING SET, we can prove a similar result. The argument for getting the following results are similar to one for MAXIMUM INDEPENDENT SET shown in the previous section, and are omitted here.

Lemma 4.4.
There exists an FPT algorithm of MINIMUM DOMINATING SET *for unit disk graphs parameterized by* α *if and only if There exists an FPT algorithm of* k-DOMINATING SET *for unit disk graphs.* □

Theorem 4.5.
MINIMUM DOMINATING SET *on unit disk graphs parameterized by area* α *is W[1]-complete.* □

5 Conclusion

We considered unit disk graphs restricted in a square with area α and consider some combinatorial problems on unit disk graphs parameterized by α. We showed HAMILTONIAN CIRCUIT and k-COLORING are both FPT, but MAXIMUM INDEPENDENT SET and MINIMUM DOMINATING SET are both W[1]-hard. Unit disk graphs on small area has wide applications, and many problems must be examined on this model.

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